Ultrasonic Attenuation in Superconductors for $ql<1$

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A derivation is given for the attenuation of both transverse and longitudinal ultrasonic waves in the superconducting state for the case when the product of the ultrasonic wave vector times the electron mean free path is smaller than one. It is assumed that the effect of electromagnetic fields is negligible in this frequency range. For transverse waves it is found that the ratio of the ultrasonic attenuation coefficient in the superconducting state and the normal state is equal to twice the Fermi function of the temperature-dependent superconducting energy gap, in agreement with experimental results obtained by Levy, Kagiwada, and Rudnick. The same result is obtained for longitudinal waves.

INTRODUCTION

FOR large values of ql, $ql>1$, where q=ultrasonic wave vector and l=electron mean free path, ultrasonic attenuation of transverse waves experiences a sharp drop when the temperature is lowered slightly below T_c, the transition temperature of the superconductor.[1] Usually, the magnitude of this drop is not equal to the total attenuation due to electron-phonon interaction in the normal state, the difference or residual attenuation gradually decreases to zero as the temperature is lowered below T_c. Morse[1] and Tsuneto[2] attribute the sharp drop to the fact that the onset of the Meissner effect screens the transverse magnetic fields proposed by Pippard[3] in his derivation of ultrasonic attenuation of transverse waves in the normal state. Morse and Claiborne[4] believe that the residual attenuation is produced by a collision drag interaction, that is, the assumption that scattering produces a distribution which is in equilibrium with the local ion motions accompanying the wave.

Recent experiments[5] on ultrasonic attenuation of shear waves in the superconducting state when $ql<1$, indicate that the ratio of the attenuation coefficient in the superconducting state and normal state, α_s/α_n, does not experience the sharp drop that is observed when $ql>1$. Moreover, α_s/α_n appears to be a similar function of the reduced temperature as is found experimentally for longitudinal waves (see Fig. 1).

This paper will concern itself mainly with ultrasonic attenuation in the superconducting state for transverse and longitudinal waves when $ql<1$. It is proposed that for transverse waves, because of the Meissner effect, no electromagnetic fields, in the sense of Pippard's fields, will be set up, and that attenuation will be produced by scattering processes only. For longitudinal waves it will be assumed that potential gradients will be set up which are produced by the density gradients which accompany a dilatational wave. Again, any electric fields that are produced by possible space charges, as postulated by Pippard for the normal state, will be neglected. Their effect will turn out to be negligible even in the normal state. With these assumptions and using Boltzmann's transport equation and the Bardeen, Cooper, and Schrieffer[6] (BCS) distribution function for the superconducting state, one may compute the rate of transfer of energy from the superconducting electrons to the lattice. This is equal to the rate-of-energy absorption by the electrons from the ultrasonic waves, and, therefore, yields the attenuation coefficient.

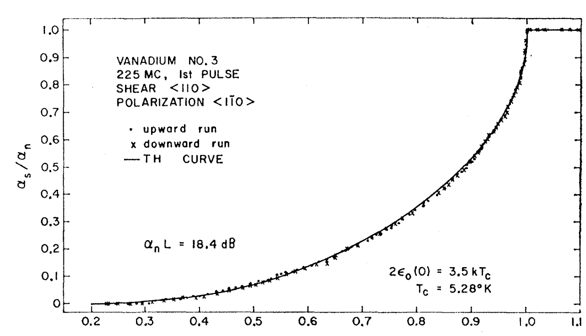

FIG. 1. Ultrasonic attenuation coefficient of transverse waves in the superconducting state for $ql<1$. The dots are experimental points obtained while the temperature of the sample was increased in discrete steps and the crosses were obtained while the temperature of the sample was decreased. $\alpha_n L$ is the total attenuation due to electron-phonon interaction that the ultrasonic pulse experiences in one round trip while the sample is in the normal state. It is a constant throughout this temperature range. The propagation direction is along ⟨110⟩. The solid line is drawn according to Eq. (8). These data were obtained from Ref. 5.

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[1] See, for instance, R. W. Morse, *Progress in Cryogenics* (Heywood and Company Ltd., London, 1959), Vol. I.

[2] T. Tsuneto, Phys. Rev. **121**, 402 (1961).

[3] A. B. Pippard, Phil. Mag. **46**, 1104 (1955). Pippard assumes that the lattice vibrations produced by ultrasonic waves deform the Fermi surface by setting up electric fields parallel to the propagation direction. These electric fields may be produced, in the case of longitudinal waves, by the fact that the small density changes of the electrons and ions are not in phase and thus space charges are produced. In the case of transverse waves, the electrons do not follow the transverse lattice motion in phase, thus an alternating magnetic field will be set up, which produces by induction an electric field in the direction of motion of the lattice.

[4] R. W. Morse, IBM J. Res. Dev. **6**, 58 (1962).

[5] M. Levy, R. Kagiwada, and I. Rudnick, in *Proceedings of the Eighth International Congress on Low Temperature Physics* (Butterworth and Co., Ltd., London, 1962).

[6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

TRANSVERSE WAVES

A method similar to the ones employed by Pippard[3] and Holstein[7] will be followed. Let us consider a transverse ultrasonic wave propagating in the z direction with a velocity v_t' and an angular frequency, ω. The wave vector is then $q=\omega/v_t'$. All variables associated with the wave are multiplied by $e^{i(\omega t-qz)}$ and, therefore,

$$\frac{\partial}{\partial t}=i\omega;\quad \frac{\partial}{\partial x}=\frac{\partial}{\partial y}=0;\quad \frac{\partial}{\partial z}=-iq.$$

Since this is a shear wave there is no density change associated with the wave. The local particle velocity assumed in the y direction will be $\mathbf{u}$. We shall also assume an average relaxation time, τ.

In order to take into account the motion of the lattice, the collision term in the Boltzmann transport equation is modified to

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}=-\frac{f-f_{\mathbf{u}}}{\tau},\qquad (1)$$

where $f_{\mathbf{u}}$ is the electron distribution corresponding to an average electron velocity equal to the local lattice velocity. The Boltzmann transport equation becomes

$$\frac{\partial f}{\partial t}+v_z\frac{\partial f}{\partial z}+a\ \text{grad}_{\mathbf{v}}f=-\frac{f-f_{\mathbf{u}}}{\tau},$$

where v_z is the z component of the electron velocity $\mathbf{v}$ and a is its acceleration which we assume to be zero since we have postulated that there are no electromagentic fields present due to the Meissner effect. Therefore,

$$\frac{\partial f}{\partial t}+v_z\frac{\partial f}{\partial z}=-\frac{f-f_{\mathbf{u}}}{\tau}.\qquad (2)$$

The distribution function for the superconducting state is given by[6]

$$f_0=1/(e^{E/kT}+1),\qquad (3)$$

where $E=[\epsilon^2+\epsilon_0^2(T)]^{1/2}$, ϵ is the energy of the normal Bloch wave referred to the Fermi level E_F and $2\epsilon_0(T)$ is the energy gap. The equilibrium value of E in the disturbed metal is given by

$$E=[(\tfrac{1}{2}m(\mathbf{v}-\mathbf{u})^2-E_F)^2+\epsilon_0^2(T)]^{1/2}.$$

Expanding $f_{\mathbf{u}}$ about its equilibrium value in the undisturbed metal f_0, one obtains

$$f_{\mathbf{u}}=f_0-\frac{mvu\sin\theta\cos\phi}{[1+\epsilon_0^2(T)/\epsilon^2]^{1/2}}\frac{\partial f_0}{\partial E},$$

where θ is the angle between the electron velocity and the propagation direction, and ϕ is the azimuthal angle, measured from the polarization direction.

[7] T. Holstein, Phys. Rev. **113**, 479 (1959).

Since one assumes that the disturbance of the distribution function is small, one may define $f=f_0+\psi$.

From Eq. (2) one obtains

$$\psi=\frac{mvu\sin\theta\cos\phi}{[1+i\omega\tau-iqv\tau\cos\theta][1+\epsilon_0^2(T)/\epsilon^2]^{1/2}}\frac{\partial f_0}{\partial E}.$$

After neglecting terms in[8] $\omega\tau$

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}=\frac{iqmv^2\tau u\sin\theta\cos\theta\cos\phi}{\tau[1-iqv\tau\cos\theta][1+\epsilon_0^2(T)/\epsilon^2]^{1/2}}\frac{\partial f_0}{\partial E}.$$

The rate of heat production due to collisions is given by[9]

$$Q=\int\left\langle H\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}\right\rangle d\mathbf{v},\qquad (4)$$

where the average is to be taken per cycle.

If one assumes phonon drag[7] then the Hamiltonian is given by

$$H=\tfrac{1}{2}m(\mathbf{v}-\mathbf{u})^2=\tfrac{1}{2}m(v^2-2vu\sin\theta\cos\phi+u^2).$$

Therefore,

$$Q=\int_{-E_f}^{\infty}\int_0^{\pi}\int_0^{2\pi}\frac{mv^3u^2\sin^3\theta\cos^2\theta\cos^2\phi}{2\tau[b^2+\cos^2\theta][1+\epsilon_0^2(T)/\epsilon^2]^{1/2}}\times\frac{\partial f_0}{\partial E}\frac{d\phi d\theta d\epsilon}{(\hbar^3/2m^3)},$$

where

$$b=\frac{1}{qv\tau}$$

and

$$d\mathbf{v}=\frac{v}{m}\frac{\sin\theta d\phi d\theta d\epsilon}{(\hbar^3/2m^3)}.$$

Approximating $-E_f$ by $-\infty$, observing that the resulting integral is even in ϵ, and that v is positive, and integrating with respect to θ and ϕ one obtains

$$Q=\frac{2\pi mu^2}{\tau(\hbar^3/2m^3)}\int_{\epsilon_0(T)}^{\infty}v^3[\tfrac{2}{3}+b^2+(b+b^3)\tan^{-1}(1/b)]\frac{\partial f_0}{\partial E}dE,$$

which yields

$$Q=\frac{mu^2N}{\tau}\left[1+\frac{3}{2(ql)^2}-\frac{3}{2}\left(\frac{1}{ql}+\frac{1}{(ql)^3}\right)\tan^{-1}ql\right]f_0(\epsilon_0),$$

where N is the density of electrons in the normal state, v_0 is the Fermi velocity, and $l=v_0\tau$.

Since the ultrasonic attenuation coefficient is given by

$$\alpha=2Q/\rho v'u^2,$$

[8] One may neglect terms in $\omega\tau$ since the electrons that will be involved in the electron phonon process are close to the Fermi level and in this case $\omega\tau=(v'/v_0)ql$, where v_0 is the Fermi velocity, which is usually about 300 times larger than the sound velocity.

[9] E. I. Blount, Phys. Rev. **114**, 418 (1959).

where ρ is the density, one obtains for transverse waves

$$\alpha_{st}=(2mN/v_t'\rho\tau)[1-g]f_0(\epsilon_0), \quad (5)$$

where the subscript t refers to transverse waves, and

$$g=\frac{3}{2(ql)^2}\left[\left(\frac{1}{ql}+\frac{1}{(ql)^3}\right)\tan^{-1}ql-1\right].$$

Finally, for $ql<1$,

$$\alpha_{st}=(2mN/5v_t'\rho\tau)(ql)^2f_0(\epsilon_0). \quad (6)$$

For the normal metal $\epsilon_0=0$ and $f_0(\epsilon_0)=\frac{1}{2}$, therefore one obtains

$$\alpha_{nt}=(mN/5v_t'\rho\tau)(ql)^2. \quad (7)$$

This result agrees with Pippard's result for $ql<1$. The ratio of the ultrasonic attenuation in the two states is therefore given by

$$\alpha_{st}/\alpha_{nt}=2f_0(\epsilon_0)=2/(e^{\epsilon_0/kT}+1). \quad (8)$$

The solid curve in Fig. 1 is plotted according to Eq. (8) using the BCS temperature dependence of the energy gap. It is plotted for a zero-temperature energy gap, $2\epsilon_0(0)$, of $3.5kT_c$.

Since neglecting the electromagnetic fields still gives the proper value for the attenuation coefficient in the normal state, one may conclude that its effect is negligible for $ql<1$. It is probable that it has a noticeable effect on the total ultrasonic attenuation in the normal state only when $ql>1$. In this instance one might expect that in the superconducting state the screening due to the Meissner effect would inhibit that part of the attenuation which would be produced mainly by the magnetic fields. Assuming this to be the case, one may find the magnitude of the drop near T_c for $ql>1$ by subtracting the value we have obtained for the whole range, neglecting the electromagnetic fields, Eq. (5), from that value obtained by Pippard for the attenuation in the normal state,

$$\alpha_{nt}=\frac{1-g}{g}\frac{Nm}{\rho v_t'\tau};$$

thus, we would obtain for the drop $\Delta\alpha=[(1-g)^2/g]\times(Nm/\rho v_t'\tau)$. This is the result obtained by Morse and Claiborne using the Boltzmann equation and the London equation. They felt that since the region of interest for the attenuation drop is very close to T_c, the superconducting behavior could be reasonably accounted for by use of the London equation. Their measurements in aluminum for $0.8\leq ql\leq 4.0$ verify this relationship. The ratio of the residual attenuation in the superconducting state to the attenuation in the normal state is

$$\alpha_{st}/\alpha_{nt}=2gf_0(\epsilon_0).$$

The ratio of the drop to the attenuation in the normal state is given by $1-g$, this is proportional to the frequency squared for small values of ql and approaches unity for large values of ql. This is consistent with Tsuneto's calculations for $ql\gg1$ which indicate that the attenuation of transverse waves should drop to a very small value near T_c. For $ql\gg1$ the residual attenuation according to Eq. (5) becomes independent of the frequency. This result should not be too surprising, since Pippard finds that this condition occurs in the normal state when the electromagnetic fields fall to a low value. However, in the normal state this happens when the skin depth becomes larger than the wavelength or when $\omega\tau$ becomes much larger than unity, while in the superconducting state it has already occurred when $ql\gg1$.

LONGITUDINAL WAVES

For longitudinal waves the Boltzmann transport equation becomes

$$\frac{\partial f}{\partial t}+v_z\frac{\partial f}{\partial z}+a\ \mathrm{grad}_v f=-\frac{f-f_{\mathbf{u},n}}{\tau}, \quad (9)$$

where $f_{\mathbf{u},n}$ is the distribution function for the electrons having an average velocity $\mathbf{u}$ in the z direction and undergoing a small density change, n.

The acceleration produced by the density gradients may be computed as follows:

$$a=-\frac{1}{m}\frac{\partial E_F}{\partial z}=\frac{iqv_0^2n}{3N}\cos\theta$$

and

$$f_{\mathbf{u},n}=f_0-\frac{mvu\cos\theta-(mnv_0^2/3N)}{[1+\epsilon_0^2(T)/\epsilon^2]^{1/2}}\frac{\partial f_0}{\partial E}.$$

Again let $f=f_0+\psi$ and, since ψ is small, $\mathrm{grad}_v\psi$ may be neglected and one obtains for ψ from Eq. (9)

$$\psi=\frac{-mvu\cos\theta+(mnv_0^2/3N)(1-iqv\tau\cos\theta)}{[1+i\omega\tau-iqv\tau\cos\theta][1+\epsilon_0^2(T)/\epsilon^2]^{1/2}}\frac{\partial f_0}{\partial E}.$$

From the continuity equation we know that the electron current density is $-env_l'$, and since there are no space charges set up it must cancel the lattice current density, eNu. Therefore, $n=Nu/v_l'$, the subscript l refers to longitudinal waves.

After neglecting terms in $\omega\tau$, we have

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}}=\frac{ium[v\cos^2\theta-(v_0^3/3v)]}{\tau[b-i\cos\theta][1+\epsilon_0^2(T)/\epsilon^2]^{1/2}}\frac{\partial f_0}{\partial E}.$$

Now the rate of heat dissipation may be computed according to Eq. (4). After integrating with respect to θ and ϕ and neglecting higher order terms

$$Q=\frac{4\pi u^2m}{\tau(\hbar^3/2m^3)}\int_{\epsilon_0(T)}^{\infty}v^2(qv\tau)^2\left(\frac{v^2}{5}-\frac{v_0^2}{9}\right)\frac{\partial f_0}{\partial E}dE,$$

which gives

$$Q=\frac{4u^2mN}{15\tau}(ql)^2f_0(\epsilon_0(T))$$

and, finally,

$$\alpha_{sl}=\frac{8mN}{15\rho v_l'\tau}(ql)^2f_0(\epsilon_0).$$

For the normal state,

$$\alpha_{nl}=\frac{4mN}{15\rho v_l'\tau}(ql)^2.$$

This attenuation agrees with Pippard's result for longitudinal waves for $ql<1$. Again the ratio is given by

$$\alpha_{sl}/\alpha_{nl}=2f_0(\epsilon_0).$$

Tsuneto has obtained the same result by using a matrix density formalism and assuming that the interaction between long-wavelength sound waves and electrons in a metal is mainly electromagnetic. A similar result was obtained by BCS for longitudinal waves for $ql>1$ by computing the net rate of absorption of energy in the superconducting state produced by direct absorption and induced emission of the imposed acoustic phonons. Since our result for the attenuation coefficient of longitudinal waves is similar to that obtained by Pippard, we may assume that the effect of the space charges may be neglected even in the normal state for $ql<1$. However, when $ql>1$, the above derivation which neglects space charges would not give the correct limit for α_n.

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Phonon Scattering by Lattice Defects*

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The Green's-function matrix method first developed by I. M. Lifshitz is applied to the problem of the scattering of phonons by a localized perturbation in the lattice. The scattering can be described by a t matrix that is localized to the same extent as the perturbation and has similar symmetry properties. The t matrix can be written in terms of the perturbation matrix γ and the Green's-function matrix g, perhaps most easily in terms of the representation formed by the eigenvectors of the matrix $g\gamma,\gamma$; these vectors can often be found by symmetry considerations. Two cases are of particular interest: (1) a "singular" perturbation which leads to a t matrix independent of the strength of the perturbation, and (2) resonance scattering from a low-frequency virtual local mode. The latter case is discussed for the example of decreased central-force constants between ⟨100⟩ nearest neighbors and the impurity site. Some implications for thermal conductivity are mentioned.

I. INTRODUCTION

In a series of papers that are as much as twenty years old, I. M. Lifshitz formally solved the dynamics of a crystal perturbed by a defect.[1–4] He assumed that the normal modes and frequencies were known for the unperturbed lattice, and by the use of the dynamic Green's-function matrix was able to reduce the number of degrees of freedom of the perturbed problem to a manageable size, essentially equal to the number of changes induced by the perturbation.

Subsequent work has been devoted mainly to one aspect of the perturbed problem, namely, the appearance of discrete frequencies belonging to lattice modes localized around the impurity.[5–7] The Green's-function matrix method may be readily applied to the electron impurity problem if Wannier functions are used, as shown by Koster and Slater.[8,9] In this case, the local modes correspond to bound electronic impurity states.

Lifshitz also discussed the problem of the remaining modes which still have running wave character.[3,4] As

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